

Generating the mapping class group of a punctured surface by involutions

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Abstract

Let $\Sigma_{g,b}$ denote a closed orientable surface of genus g with b punctures and let $\text{Mod}(\Sigma_{g,b})$ denote its mapping class group. In [Luo] Luo proved that if the genus is at least 3, $\text{Mod}(\Sigma_{g,b})$ is generated by involutions. He also asked if there exists a universal upper bound, independent of genus and the number of punctures, for the number of torsion elements/involutions needed to generate $\text{Mod}(\Sigma_{g,b})$. Brendle and Farb [BF] gave an answer in the case of $g \geq 3, b = 0$ and $g \geq 4, b = 1$, by describing a generating set consisting of 6 involutions. Kassabov showed that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. We proved that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 7$ and 5 involutions if $g \geq 5$.

1 Introduction

Let $\Sigma_{g,b}$ be a closed orientable surface of genus $g \geq 1$ with arbitrarily chosen b points (which we call punctures). Let $\text{Mod}(\Sigma_{g,b})$ be the mapping class group of $\Sigma_{g,b}$, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. Let $\text{Mod}^\pm(\Sigma_{g,b})$ be the extended mapping class group of $\Sigma_{g,b}$, which is the group of homotopy class of all (including orientation-reversing) homeomorphisms preserving the set of punctures. By $\text{Mod}_{g,b}^0$ we will denote the subgroup of $\text{Mod}_{g,b}$ which fixes the punctures pointwise. It is clear that we have the exact sequence:

$$1 \rightarrow \text{Mod}_{g,b}^0 \rightarrow \text{Mod}_{g,b} \rightarrow \text{Sym}_b \rightarrow 1,$$

where the last projection is given by the restriction of a homeomorphism to its action on the puncture points.

The study of the generators for the mapping class group of a closed surface was first considered by Dehn. He proved in [De] that $\text{Mod}(\Sigma_{g,0})$ is generated by a finite set of Dehn twists. Thirty years later, Lickorish [Li] showed that $3g - 1$ Dehn twists generate $\text{Mod}_{g,0}$. This number was improved to $2g + 1$ by Humphries [Hu]. Humphries proved, moreover, that in fact the number $2g + 1$ is minimal; i.e. $\text{Mod}(\Sigma_{g,0})$ cannot be generated by $2g$ (or less) Dehn twists. Johnson [Jo] proved that the $2g + 1$ Dehn twists also generate $\text{Mod}(\Sigma_{g,1})$. In the case of multiple punctures the mapping class group can be generated by $2g + b$ Dehn twists for $b \geq 1$ (see [Ge]).

It is possible to obtain smaller generating sets of $\text{Mod}(\Sigma_{g,b})$ by using elements

other than twists. N.Lu (see [Lu]) constructed a generated set of $\text{Mod}(\Sigma_{g,0})$ consisting of 3 elements. This result was improved by Wajnryb who found the smallest possible generating set of $\text{Mod}(\Sigma_{g,0})$ consisting of 2 elements (see [Wa]). Korkmaz proved in [Ko] that one of these generators can be taken as a Dehn twist. It is also known that in the case of $b = 0$ the mapping class group can be generated by 3 torsion elements (see [BF]). More, Korkmaz showed in [Ko] that the mapping class group can be generated by 2 torsion elements (also in the case of $b = 0, 1$). In [Ma], Maclachlan proved that the moduli space is simply connected as a topological space by showing that $\text{Mod}(\Sigma_{g,0})$ is generated by torsion elements. Several years later Patterson generalized these results to $\text{Mod}(\Sigma_{g,b})$ for $g \geq 3, b \geq 1$ (see [Pa]).

In [MP], McCarthy and Papadopoulos proved that $\text{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugates of a single involution for $g \geq 3$. Luo, see [Luo], described the finite set of involutions which generate $\text{Mod}(\Sigma_{g,b})$ for $g \geq 3$. He also proved that $\text{Mod}(\Sigma_{g,b})$ is generated by torsion elements in all cases except $g = 2$ and $b = 5k + 4$, but this group is not generated by involutions if $g \leq 2$. Brendle and Farb proved that $\text{Mod}(\Sigma_{g,b})$ can be generated by 6 involutions for $g \geq 3, b = 0$ and $g \geq 4, b \leq 1$ (see [BF]). In [Ka], Kassabov proved that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. He also proved in the case of $\text{Mod}^\pm(\Sigma_{g,b})$. Our main result is stronger than [Ka].

Main Theorem. *For all $g \geq 3$ and $b \geq 0$, the mapping class group $\text{Mod}(\Sigma_{g,b})$ can be generated by:*

- (a) 4 involutions if $g \geq 7$;
- (b) 5 involutions if $g \geq 5$.

2 Preliminaries

Let c be a simple closed curve on $\Sigma_{g,b}$. Then the (right hand) Dehn twist T_c about c is the homotopy class of the homeomorphism obtained by cutting $\Sigma_{g,b}$ along c , twisting one of the side by 360° to the right and gluing two sides of a back to each other. Figure 1 shows the Dehn twist about the curve c . We will

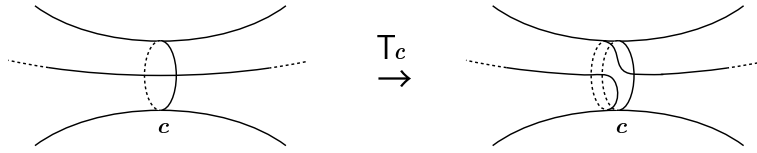


Figure 1: The Dehn twist

denote by T_c the Dehn twist around the curve c .

We record the following lemmas.

Lemma 1. *For any homeomorphism h of the surface $\Sigma_{g,b}$ the twists around the curves c and $h(c)$ are conjugate in the mapping class group $\text{Mod}(\Sigma_{g,b})$,*

$$T_{h(c)} = hT_ch^{-1}.$$

Lemma 2. *Let c and d be two simple closed curves on $\Sigma_{g,b}$. If c is disjoint from d , then*

$$T_c T_d = T_d T_c$$

3 Proof of main theorem

In this section we proof maintheorem. The keypoints of proof are to generate T_α in 4 involutions by using lantern relation.

3.1 The policy of proof

We give the policy of proof of maintheorem.

Lemma 3. *Let G, Q denote the groups and let N, H denote the subgroups of G . Suppose that the group G has the following exact sequence;*

$$1 \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \rightarrow 1.$$

If H contains $i(N)$ and has a surjection to Q then we have that $H = G$.

Proof. We suppose that there exists some $g \in G - H$. By the existence of surjection from H to Q , we can see that there exists some $h \in H$ such that $\pi(h) = \pi(g)$. Therefore, since $\pi(g^{-1}h) = \pi(g)^{-1}\pi(h) = 1$, we can see that $g^{-1}h \in \text{Ker } \pi = \text{Im } i$. Then there exists some $n \in N$ such that $i(n) = g^{-1}h$. By $i(N) \subset H$, since $i(n) \in H$ and $h \in H$, we have

$$g = h \cdot i(n)^{-1} \in H.$$

This is contradiction in $g \notin H$. Therefore, we can prove that $H = G$. □

It is clear that we have the exact sequence:

$$1 \rightarrow \text{Mod}_{g,b}^0 \rightarrow \text{Mod}_{g,b} \rightarrow \text{Sym}_b \rightarrow 1.$$

Therefore, we can see the following corollary;

Corollary 4. *Let H denote the subgroup of $\text{Mod}(\Sigma_{g,b})$, which contains $\text{Mod}^0(\Sigma_{g,b})$ and has a surjection to Sym_b . Then H is equal to $\text{Mod}(\Sigma_{g,b})$.*

We generate the subgroup H which has the condition of corollary 4 by involutions.

Let us embed our surface $\Sigma_{g,b}$ in the Euclidian space in two different ways as shown on Figure 2. (In these pictures we will assume that genus $g = 2k + 1$ is odd and the number of punctures $b = 2l + 1$ is odd. In the case of even genus we only have to swap the top parts of the pictures, and in the case of even number of punctures we have to remove the last point.)

In Figure 2 we have also marked the puncture points as x_1, \dots, x_b and we have the curves $\alpha_i, \beta_i, \gamma_i$ and δ . The curve $\alpha_i, \beta_i, \gamma_i$ are non separating curve and δ is separating curve.

Each embedding gives a natural involution of the surface—the half turn rotation around its axis of symmetry. Let us call these involutions ρ_1 and ρ_2 .

Then we can get following lemma;

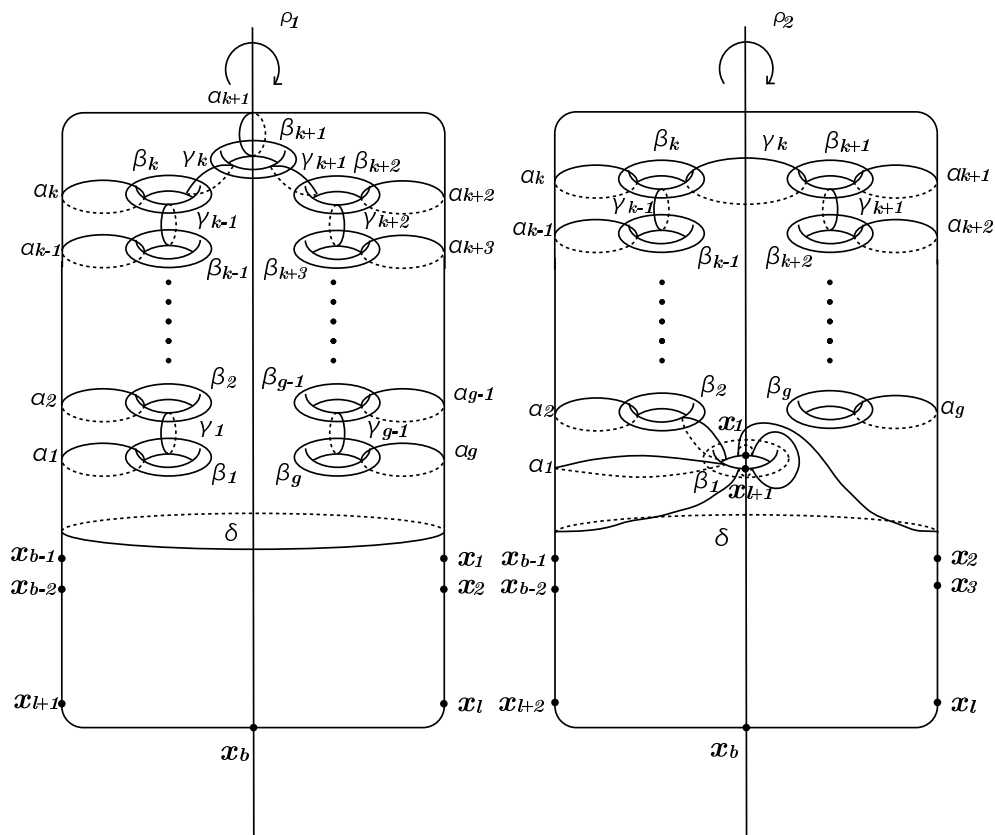


Figure 2: The embeddings of the surface $\Sigma_{g,b}$ in Euclidian space used to define the involutions ρ_1 and ρ_2 .

Lemma 5. *The subgroup of the mapping class group be generated by ρ_1 , ρ_2 and 3 Dehn twists T_α , T_β and T_γ around one of the curve in each family contains the subgroup $\text{Mod}^0(\Sigma_{g,b})$.*

We postpone the proof of lemma 5 until Section 5.

The existence a surjection from the subgroup H of $\text{Mod}(\Sigma_{g,b})$ to Sym_b is equivalent to showing that the Sym_b can be generated by involutions;

$$\begin{aligned} r_1 &= (1, b-1)(2, b-2) \cdots (l, l+1)(b) \\ r_2 &= (2, b-1)(3, b-2) \cdots (l, l+2)(1)(l+1)(b) \\ r_3 &= (1, b)(2, b-1)(3, b-2) \cdots (l, l+2)(l+1) \end{aligned}$$

corresponding to 3 involutions in H .

Lemma 6. *The symmetric group Sym_b is generated by r_1, r_2 and r_3 .*

Proof. The group generated by r_i contains the long cycle $r_3 r_1 = (1, 2, \dots, b)$ and transposition $r_3 r_2 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions r_i generate Sym_b . \square

We note that the images of ρ_1 and ρ_2 to Sym_b are r_1 and r_2 .

Therefore, by Lemma 1, Corollary 4, Lemma 5 and Lemma 6 we sufficient to generate H by ρ_1 , ρ_2 and involutions which have the following conditions;

- $\langle 1 \rangle$ involutions which generate the Dehn twist around γ ,
- $\langle 2 \rangle$ two of each involutions which exchange α and β , β and γ , γ and α ,
- $\langle 3 \rangle$ involution whose image is r_3 .

3.2 Generating Dehn twists by 4 involutions

In this subsection, we argue about $\langle 1 \rangle$. Moreover, we generate Dehn twists by 4 involutions. The basic idea is to use the lantern relation.

We begin by recalling the lantern relation in the mapping class group. This relation was first discovered by Dehn and later rediscovered by Johnson.

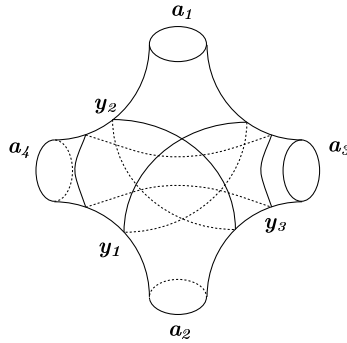


Figure 3: Lantern

From now on we will assume that the genus g of the surface is at least 5. Let the $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by a_1, a_2, a_3 and a_4 the four boundary curves of the surface $S_{0,4}$ and let the interior curves y_1, y_2 and y_3 be as shown in Figure 3. The following relation:

$$T_{y_1}T_{y_2}T_{y_3} = T_{a_1}T_{a_2}T_{a_3}T_{a_4}. \quad (1)$$

among the Dehn twists around the curves a_i and y_i is known as the lantern relation. Notice that the curves a_i do not intersect any other curve and that the Dehn twists T_{a_i} commute with every twists in this relation. This allows us to rewrite the lantern relation as follows

$$T_{a_4} = (T_{y_1}T_{a_1}^{-1})(T_{y_2}T_{a_2}^{-1})(T_{y_3}T_{a_3}^{-1}). \quad (2)$$

Let R denote the product $\rho_2\rho_1$. By Figure 2 we can see that $R = \rho_2\rho_1$ acts as follows:

$$\begin{aligned} R\alpha_i &= \alpha_{i+1}, \quad (1 \leq i < g) \\ R\beta_i &= \beta_{i+1}, \quad (1 \leq i < g) \\ R\gamma_i &= \gamma_{i+1}, \quad (1 \leq i < g-1). \end{aligned} \quad (3)$$

The lanterns S and $R^{-2}S$ have a common boundary component $a_1 = R^{-2}a_2$ and their union is a surface S_2 homeomorphic to a sphere with 6 boundary components. By Figure 4 we can see that there exists an involution \tilde{J} of S_2 which takes S to $R^{-2}S$.

Let us embed the surface S_2 in $\Sigma_{g,b}$ as shown on Figure 5. The boundary components of S_2 are $a_1 = \alpha_k, a_2 = \alpha_{k+2}, a_3 = \gamma_{k+1}, a_4 = \gamma_k, R^{-2}a_1 = \alpha_{k-2}, R^{-2}a_2 = \alpha_k, R^{-2}a_3 = \gamma_{k-1}$ and $R^{-2}a_4 = \gamma_{k-2}$; and the middle curve $y_1 = \alpha_{k+1}$. The Figure 5 shows the existence of the involution \tilde{J} on the complement of S_2 which is a surface of genus $g-5$ with 6 boundary components. Gluing together \tilde{J} and \tilde{J} gives us the involution J of the surface $\Sigma_{g,b}$. By Figure 4 J acts as follows

$$J(a_1) = R^{-2}a_2, \quad J(a_3) = R^{-2}a_1, \quad J(y_1) = R^{-2}y_2, \quad J(y_3) = R^{-2}y_1.$$

Therefore, we have

$$\begin{aligned} R^2J(a_1) &= a_2, \quad R^2J(y_1) = y_2 \\ JR^{-2}(a_1) &= a_3, \quad JR^{-2}(y_1) = y_3. \end{aligned} \quad (4)$$

Let ρ_3 denote $T_{a_1}\rho_2T_{a_1}^{-1}$.

By Lemma 1, (4) and that ρ_2 sends $a_1 = \alpha_k$ to $y_1 = \alpha_{k+1}$, we have

$$\begin{aligned} T_{y_1}T_{a_1}^{-1} &= \rho_2T_{a_1}\rho_2T_{a_1}^{-1} = \rho_2\rho_3, \\ T_{y_2}T_{a_2}^{-1} &= R^2J\rho_2\rho_3JR^{-2}, \quad T_{y_3}T_{a_3}^{-1} = JR^{-2}\rho_2\rho_3R^2J. \end{aligned} \quad (5)$$

By (2) and (5) we have

$$T_{\gamma_k} = (\rho_2\rho_3)(R^2J\rho_2\rho_3JR^{-2})(JR^{-2}\rho_2\rho_3R^2J). \quad (6)$$

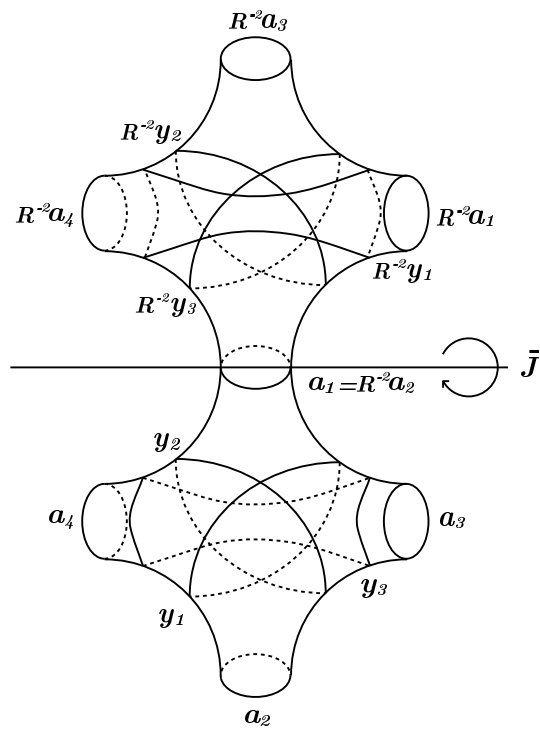


Figure 4: S_2 and the involution \bar{J}

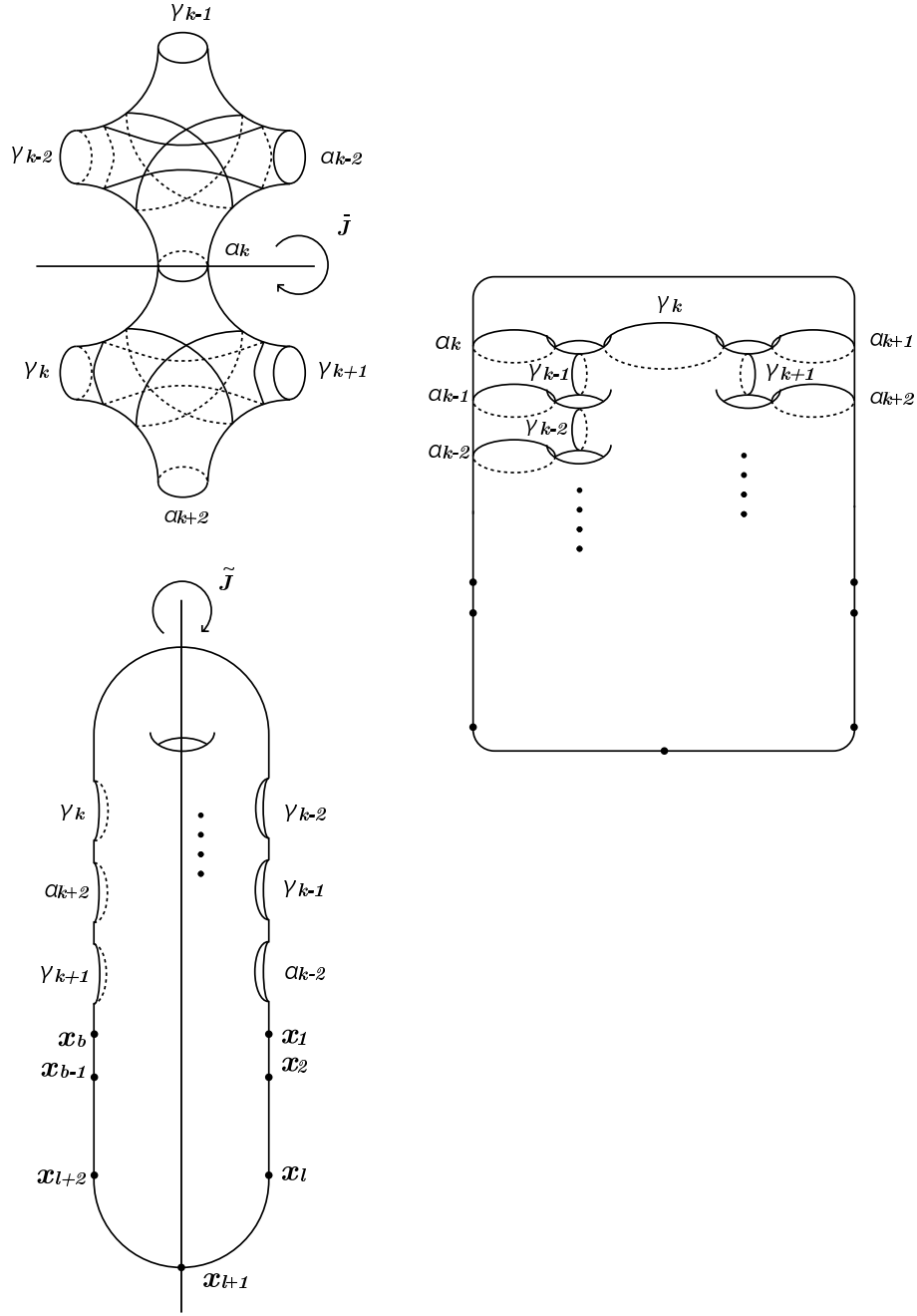


Figure 5: The involution J on $\Sigma_{g,b}$

3.3 Genus at least 5

We prove that the mapping class group is generated by 5 involutions. The five involutions are $\rho_1, \rho_2, \rho_3, J$ and another involution I . We construct involution I in the same way as involution J like Figure 6.

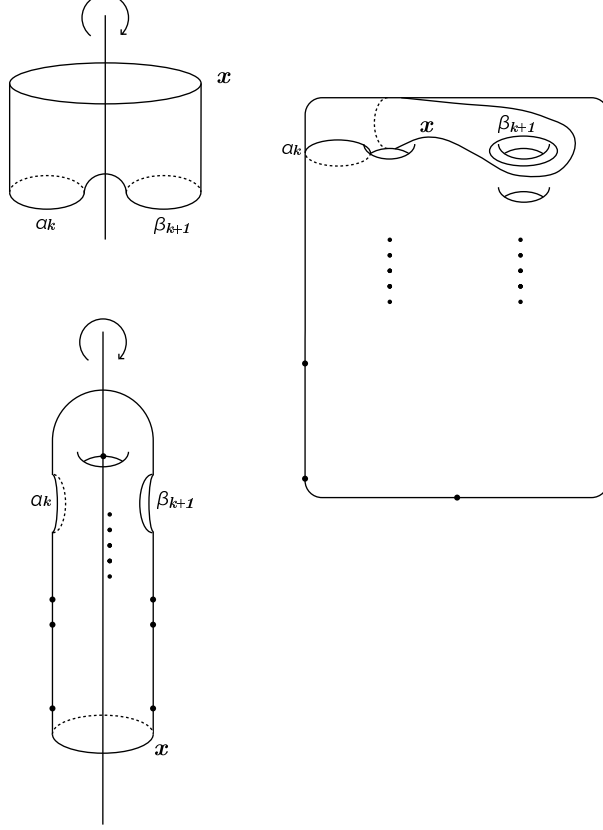


Figure 6: The involution I on $\Sigma_{g,b}$

Theorem 7. *If $g \geq 5$, the group G_3 generated by $\rho_1, \rho_2, \rho_3, I$ and J is the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.*

Proof. By the relation (6) we satisfy the condition $\langle 1 \rangle$. Since J sends α_{k-2} to γ_{k+1} and I sends α_k to β_{k+1} , we consist the condition $\langle 2 \rangle$. We can also see that we satisfy the condition $\langle 3 \rangle$ from a way to the construction of the involution J . Therefore, we can finish the proof of the theorem because we can satisfy the conditions in 3.1. \square

3.4 Genus at least 7

We want to improve the above argument and show that for the genus $g \geq 7$ we do not need the involution I in order to generate the mapping class group. Assume that the genus of the surface is at least 7.

The S_2 and two pairs of pants have common boundary components $R^{-2}a_1$ and a_3 and their union is a surface S_3 homeomorphic to a sphere with 8 boundary components. Figure 7 shows the existence of the involution \tilde{J}' on S_3 which extends the involution \tilde{J} on S_2 .

Let us embed S_3 in the $\Sigma_{g,b}$ as shown on Figure 7. From Figure 7 we can find the involution \tilde{J}' of the complement of S_3 . Let J' be the involution obtained by gluing together \tilde{J} and \tilde{J}' . Moreover, from Figure 7 we can construct J' which acts on the punctures as the involution r_3 .

Theorem 8. *If $g \geq 7$, the group G_4 generated by ρ_1 , ρ_2 , ρ_3 and J' is the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.*

Proof. From the construction of J' we have

$$T_{\gamma_k} = (\rho_2 \rho_3)(R^2 J' \rho_2 \rho_3 J' R^{-2})(J' R^{-2} \rho_2 \rho_3 R^2 J') \in G_4.$$

Therefore, we can see that we satisfy the condition $\langle 1 \rangle$. Since J' can send α_{k-2} to γ_{k+1} and β_{k+3} to γ_{k-3} , we can satisfy the condition $\langle 2 \rangle$ only in J' . Moreover, By that J' acts as r_3 , we consist the condition $\langle 3 \rangle$. Therefore, the group G_4 is the whole mapping class group. \square

4 The subgroup generated by 2 involutions and 3 Dehn twists, which contains $\text{Mod}^0(\Sigma_{g,b})$

In this section we prove Lemma 5 (i.e. we construct the subgroup of the mapping class group $\text{Mod}(\Sigma_{g,b})$ by 2 involutions and 3 Dehn twists, which contains the pure mapping class group $\text{Mod}^0(\Sigma_{g,b})$).

We recall that $R = \rho_2 \rho_1$. By Lemma 1 and (3), we get following relation;

$$\begin{aligned} T_{\alpha_{i+1}} &= RT_{\alpha_i} R^{-1} \\ T_{\beta_{i+1}} &= RT_{\beta_i} R^{-1} \\ T_{\gamma_{i+1}} &= RT_{\gamma_i} R^{-1}. \end{aligned} \tag{7}$$

Let the subgroup G of the mapping class group be generated by ρ_1 , ρ_2 and 3 Dehn twists T_{α} , T_{β} and T_{γ} around one of the curve in each family. By relation (7), $T_{\alpha_i}, T_{\beta_i}, T_{\gamma_i} \in G$ for all i .

Our next step is to show that G contains $\text{Mod}^0(\Sigma_{g,b})$. Let denote the curves $\delta', \eta', \delta'', \eta'', \delta_j, \eta_j (j = 1, \dots, l-1, l+1, \dots, b-2)$ in Figure 8. In [Ge] it is shown that $\text{Mod}^0(\Sigma_{g,b})$ is generated by Dehn twists around the curves α_i -es, β_i -es, γ_i -es, δ' , δ'' and δ_j -es, for $j = 1, \dots, l-1, l+1, \dots, b-2$.

Lemma 9. $R^{-1}(\delta_j) = \eta_{j-1}$ ($l+2 \leq j \leq b-1$), $R^{-1}(\delta_{l+1}) = \eta'$.

Proof. Figure 9 and Figure 10 shows the action of ρ_1 and ρ_2 on the curve δ' and δ_j ($j = l-1, \dots, b-1$). It is clear from the picture that $\eta_{j-1} = \rho_1 \rho_2(\delta_j) = R^{-1}(\delta_j)$. It is also showed that $R^{-1}(\delta_{l+1}) = \eta'$. \square

Lemma 10. $T_{\delta_j}, T_{\delta'}, T_{\delta''} \in G$ ($j = 1, \dots, l-1, l+1, \dots, b-2$).

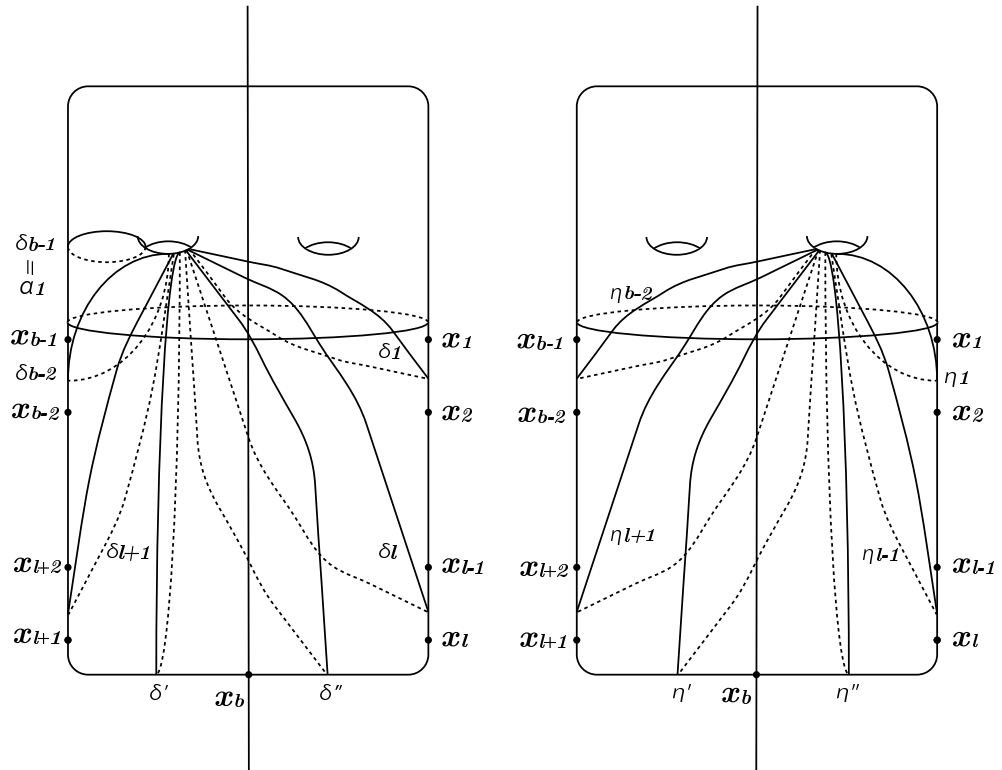


Figure 8: The curves δ_i -es, η_i -es.

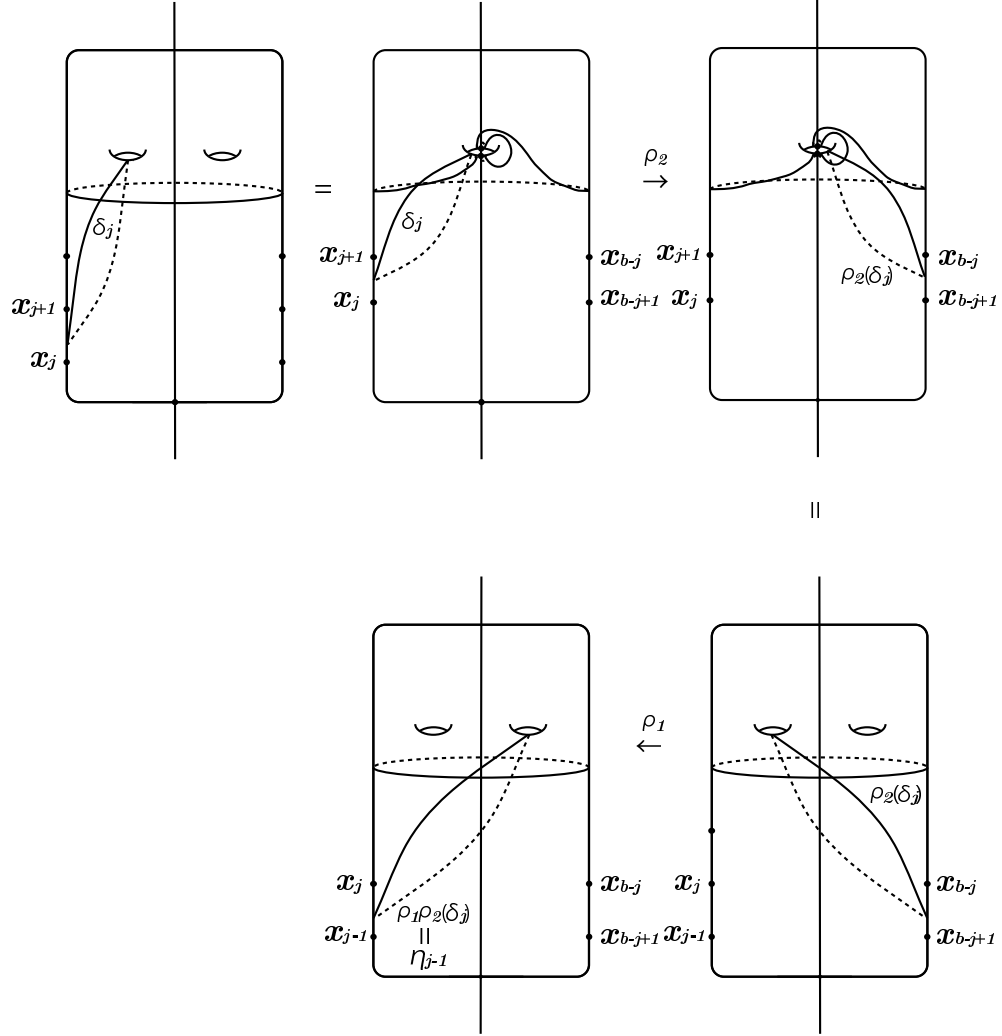


Figure 9: The action of R on the curve δ_j ($l+2 \leq j \leq b-1$)

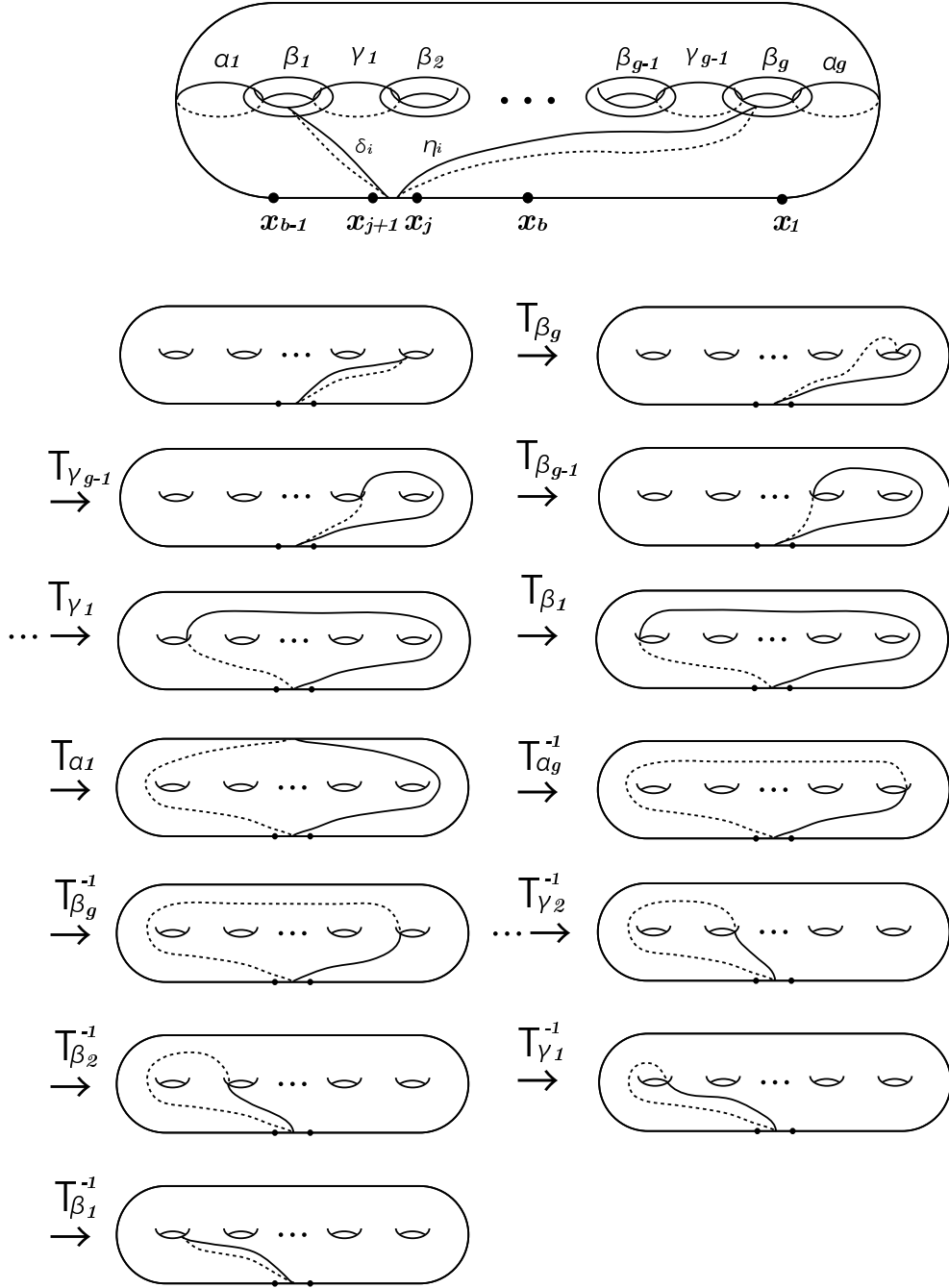


Figure 11: The action of U on the curve δ_i .

Proof. At first, we prove $T_{\delta_j} \in G$ using induction j ($j = l + 2, \dots, b - 1$) and $T_{\delta'} \in G$, then we prove that $T_j \in G$ ($j = 0, \dots, l - 1$) and $T_{\delta''} \in G$. The base case, $j = b - 1$, is clear because by construction G contains $T_{\delta_{b-1}} = T_{\alpha_1}$. Suppose that G contains the twist T_{δ_j} . Using Lemma 1 and Lemma 8 we can see that the twists $T_{\eta_{j-1}}$ and $T_{\eta'}$ also lies in G , since it is conjugate to T_{δ_j} and $T_{\delta_{l+1}}$

$$T_{\eta_{j-1}} = R^{-1}T_{\delta_j}R \in G. \quad (8)$$

Let U denote the product

$$U = T_{\beta_1}^{-1}T_{\gamma_1}^{-1}T_{\beta_2}^{-1} \dots T_{\beta_{g-1}}^{-1}T_{\gamma_{g-1}}^{-1}T_{\beta_g}^{-1}T_{\alpha_g}^{-1}T_{\alpha_1}T_{\beta_1}T_{\gamma_1}T_{\beta_2} \dots T_{\beta_{g-1}}T_{\gamma_{g-1}}T_{\beta_g} \in G.$$

The Figure 11 shows that

$$\begin{aligned} U(\eta') &= \delta' \\ U(\eta'') &= \delta'' \\ U(\eta_k) &= \delta_j \quad (j = 1, \dots, l - 1, l + 1, \dots, b - 2). \end{aligned} \quad (9)$$

By Lemma 1, Lemma 10 and (9) we can see that $T_{\delta_{j-1}} = UT_{\eta_{j-1}}U^{-1} \in G$. Thus $T_{\delta_j} \in G$ ($j = l + 2, \dots, b - 1$). Moreover, we can see that

$$R^{-1}(\delta_{l+2}) = \eta_{l+1}, U(\eta_{l+1}) = \delta_{l+1}, R^{-1}(\delta_{l+1}) = \eta', U(\eta') = \delta'.$$

Therefore, we have that $T_{\delta_{l+1}}, T_{\delta'} \in G$.

The next step we will prove that $T_{\delta''}, T_{\delta_j} \in G$ ($j = 1, \dots, l - 1$).

By Figure 3 we can see that $\rho_1(\delta'') = \eta', \rho_1(\delta_j) = \eta_{b-1-j}$ ($1 \leq j \leq l - 1$). By (9) we can understand that $T_{\delta_j} = U^{-1}T_{\eta_{b-1-j}}U, T_{\delta''} = U^{-1}T_{\eta''}U \in G$. We finished proving Lemma 10. \square

Corollary 11. *The group G contains the subgroup $\text{Mod}^0(\Sigma_{g,b})$.*

Therefore, we can prove Lemma 5.

Remark 12. *The group G is not the whole group $\text{Mod}^0(\Sigma_{g,b})$ (if $b > 3$), because its image in the symmetric group Sym_b is generated by two involutions and therefore is a dihedral group. It is easy see that this image is the group D_{2b} which is a proper subgroup of Sym_b .*

5 Remark

Clearly $\text{Mod}(\Sigma_{g,b})$ is never generated by two involutions, for then it would be a quotient of the infinite dihedral group, and so would be virtually abelian. Since the current known bounds are so close to being sharp, it is natural to ask for the sharpest bounds.

Problem. *For each $g \geq 3$, prove sharp bounds for the minimal number of involutions required to generate $\text{Mod}(\Sigma_{g,b})$. In particular, for $g \geq 7$ determine whether or not $\text{Mod}(\Sigma_{g,b})$ is generated by 3 involutions.*

In order to generate the extended mapping class group $\text{Mod}^\pm(\Sigma_{g,b})$, it suffices to add one more generator, namely the isotopy class of any orientation-reversing diffeomorphism. Therefore, by replacing the involution ρ_1 with the reflection, we can get following result;

Corollary 13. *For all $g \geq 3$ and $b \geq 0$, the extended mapping class group $\text{Mod}^\pm(\Sigma_{g,b})$ can be generated by:*

- (a) *4 involutions if $g \geq 7$;*
- (b) *5 involutions if $g \geq 5$.*

But in the case of $b = 0$, Stukow showed the result that was stronger than Corollary 12. He proved that $\text{Mod}^\pm(\Sigma_{g,0})$ is generated by three involutions. Then, we can consider following problem;

Problem. *Can the extended mapping class group $\text{Mod}^\pm(\Sigma_{g,b})$ be generated by 3 involutions ?*

6 Acknowledgement

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